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A New Theory of Shock Dynamics
Part II: Numerical Solution

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Abstract. A new theory for the calculation of the successive positions of a shock was suggested in Part I. For one dimensional problems, it required integration of a finite number (say, 4 or 5) of ordinary differential equations derived from an infinite set of compatibility conditions. Numerical results here show that the error can be made less than 0.1 percent.

1. INTRODUCTION

In Part I of this paper, the analytic considerations of this proposed theory of shock dynamics were discussed with reference to the equation $u_t + uu_x = 0$. An infinite system of ordinary differential equations was obtained for the determination of the shock position $X(t)$, the shock strength $u_0(t)$ and the spatial derivatives $u_i(t)$, $i \geq 1$. If in the $(n+2)$th equation of this system, u_{n+1} were set equal to zero, then these $(n+2)$ equations form a closed system from which the approximate shock position, $\bar{X}(t)$, the approximate shock strength $\bar{u}_0(t)$ and approximate spatial derivatives $\bar{u}_i(t)$, $i = 1, 2, \ldots n$, can be determined. The advantage of this truncated system is that it reduces the infinite dimensional problem to a finite dimensional one for which the solution is easily obtained by numerical integration.

2. EXAMPLES

The truncated system is a system of $(n+2)$ equations in $\bar{X}$, $\bar{u}_0$, $\bar{v}_i \equiv (\frac{\bar{u}_i}{i!})$

$$\frac{d\bar{X}_0}{dt} = \frac{\bar{u}_0}{2} \tag{2.1}$$

$$\frac{d\bar{u}_0}{dt} = -\frac{1}{2}\bar{u}_0\bar{v}_1 \tag{2.2}$$

$$\frac{d\bar{v}_i}{dt} = -\frac{(i+1)}{2}\bar{u}_0\bar{v}_{i+1} - \frac{(i+1)}{2}\sum_{j=1}^{i}\bar{v}_j\bar{v}_{i-j+1} \tag{2.3}$$

$$i = 1, 2, \ldots n-1$$

$$\frac{d\bar{v}_n}{dt} = -\frac{n+1}{2}\sum_{j=1}^{n}\bar{v}_j\bar{v}_{n-j+1} \tag{2.4}$$

with the following initial conditions

$$\bar{X}(0) = X_0, \text{ given}$$

$$\bar{u}_0(0) = \Phi(X_0 - 0), \bar{v}_i(0) = \frac{1}{i!}\left(\frac{d^i\Phi}{dx^i}\right)_{x=x_0-0}, i = 1, 2, \ldots n \tag{2.5}$$

X_0 is the position of the shock in the initial data $u(x,0) = \Phi(x)$.

For numerical computation, two special cases are considered.

Case 1.

$$\Phi_1(x) = \left(\frac{x+\eta}{\eta+1}\right)^2, \; x \in (-\eta, 1)$$
$$= 0 \qquad \text{elsewhere}$$

Case 2.

$$\Phi_2(x) = \alpha e^{\beta x}, \; x < 0$$
$$= 0 \quad \text{elsewhere}$$

$$\text{For } \Phi_1(x), \; X_0 = 1 \text{ and } \bar{v}_i(0) = 0 \text{ for } i \geq 3$$

Without loss of generality for $\Phi_2(x)$, we can set $\alpha = 1, \beta = 1$ because by a change of variable

$$x = \beta x, t = \frac{\beta t}{\alpha}, u = \alpha u,$$

this can be obtained.

$$\text{Here } X_0 = 0 \text{ and } \bar{u}_0(0) = \bar{v}_i(0) = 1 \text{ for all } i \geq 1$$

The exact position of the shock in this simple case can be obtained from the equation:

$$t\Phi^2(\xi) + \int_{X_0}^{\xi} \Phi(\mu)d\mu = 0,$$

where $\xi = X - u_0 t$. In the case 1 we have

$$t\left(\frac{\xi+\eta}{\eta+1}\right)^4 + \frac{2}{3}\frac{(\xi+\eta)^3 - (1+\eta)^3}{(1+\eta)^2} = 0$$

and in case 2

$$te^{2\xi} + 2(e^{\xi} - 1) = 0$$

which give the value of ξ. This in turn helps to evaluate the other quantities on the shock

$$u_0 = \Phi(\xi), v_1 = \frac{\Phi'(\xi)}{1 + t\Phi'(\xi)}$$

$$v_2 = \frac{\Phi''(\xi)}{2[1 + t\Phi'(\xi)]^3}, v_3 = \frac{\Phi'''(\xi)}{6[1 + t\Phi'(\xi)]^4} - \frac{t[\Phi''(\xi)]^2}{2[1 + t\Phi'(\xi)]^5}$$

and so on. In case 1 where $\Phi^i(\xi) = 0$ for $i > 3$, we have

$$v_i(t) = \frac{(-1)^i}{i!} \cdot \frac{1.3.5.7\ldots(2i-3)[\Phi''(\xi)]^{i-1}t^{i-2}}{[1 + t\Phi'(\xi)]^{2i-1}}, i \geq 2$$

$v_i(t) \to 0$ as $t \to \infty$ for all i and gives a value for comparison with the approximate results. The initial data in case 1 is nonzero only in the interval $(-\eta, 1)$, with a shock at $x = 1$. As $\eta \to \infty$ the disturbance behind the shock approaches a constant state $u = 1$. For η close to -1, Φ increases very rapidly from 0 to 1 over a very short distance, so that the spatial derivatives of u play an important role in this case. For numerical computation, we have chosen $\eta = -0.5$. The initial data in case 2 is not of compact support.

Tables 1 and 2 give the values of u for initial values corresponding to cases 1 and 2 at $t = 1.0, 5.0$ and 10.0. $n = k$ denotes that v_{k+1} is set equal to 0 and $k+2$ equations (2.1)–(2.4) with (2.5) are considered. For $n = 1$, the error in u is sizeable, but for $n = 2$, the error drops rapidly (less than 1% in case 2), while for $n = 3$, it is uniformly very small, as for $n = 5, 8, 25$ as well. Computation was done for the case 2 with $\alpha = 1$, $\beta = 1$.

TABLE 1: $\eta = -0.5$

	$t = 1.0$		$t = 5.0$		$t = 10.0$	
	u	% error	u	% error	u	% error
Exact	.47390445	–	.24231081	–	.17572092	–
$n = 1$	.44721360	-5.6	.21821789	-9.9	.15617376	-11.0
$n = 2$	.47171239	$-.46$	.23787367	-1.8	.17140803	-2.5
$n = 3$	.47366942	$-.50 \times 10^{-1}$	.24120493	$-.46$	.17440411	$-.75$
$n = 5$	.47390183	$-.56 \times 10^{-3}$	.24221465	$-.40 \times 10^{-1}$	.17554484	$-.10$
$n = 8$	.47390560	$.24 \times 10^{-3}$	.24230783	$-.12 \times 10^{-2}$	.17570887	$-.68 \times 10^{-2}$
$n = 25$	.47390561	$.24 \times 10^{-3}$	.24231136	$.24 \times 10^{-3}$	.17572129	$.22 \times 10^{-3}$

TABLE 2: $\Phi = e^x,\ x < 0;\ \Phi = 0;\ x \geq 0$

	$t = 1.0$		$t = 5.0$		$t = 10.0$	
	u	% error	u	% error	u	% error
Exact	.73205081	–	.46332497	–	.35825757	–
$n = 1$	.70710678	-3.4	.40824829	-12	.30151134	-16
$n = 2$	.73372900	.23	.46777169	.96	.36157950	.93
$n = 3$	.73200502	$-.63 \times 10^{-2}$	.46355666	$.50 \times 10^{-1}$	.35872978	.13
$n = 5$	.73205096	$.26 \times 10^{-4}$	.46331988	$-.11 \times 10^{-2}$	.35825020	$-.21 \times 10^{-2}$
$n = 8$	.73205081	0	.46332497	0	.35825765	$.22 \times 10^{-4}$
$n = 25$	.73205081	0	.46332496	0	.35825757	0

3. CONCLUSION

For the equation $u_t + uu_x = 0$, with initial condition having a single jump discontinuity, the shock position $X(t)$, shock strength $u_0(t)$ and the spatial derivatives at the shock $u_i(t)$ are given by an infinite set of ordinary differential equations. If this set is truncated at any stage, involving 5 or more equations, this closed system of equations can be easily integrated numerically to give very accurate results. This is in sharp contrast to the characteristic rule, where only two equations are considered and the error involved could be extremely large.

REFERENCES

1. R. Ravindran and P. Prasad, A New Theory of Shock Dynamics Part I: Analytic Considerations, *Applied Mathematics Letters* 3 (2), 77–81 (1990).

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